Dynamics of a System of N Atoms Interacting With a Radiation Field

E. R. BULEY AND F. W. CUMMINGS*
Aeronutronic Division, Philco Corporation, Newport Beach, California

(Received 4 December 1963)

The time-dependent behavior of an idealized laser is analyzed. The analysis avoids the use of the rate equations and includes N-particle "superradiant" effects. Perturbation theory is not used; the dynamics of N two-level atoms interacting strongly with a single-radiation field mode yields four coupled nonlinear equations which are integrated numerically for some special cases appropriate to lasers operating under ordinary power output as well as "Q-spoiling" conditions. Loss effects are grafted onto the dynamical equations phenomenologically. It is seen that under appropriate conditions, the laser is expected to emit one or more appreciable subsequent output bursts following an initial giant pulse, and that the subsequent emissions may be emitted by a population which is below inversion.

I. INTRODUCTION

THE continuing refinement of experimental techniques as well as of crystal growing techniques allows the mechanism underlying the production of coherent bursts of radiation, such as produced in lasers, to be probed in more detail. In the past, the analysis of the time dependence of a system of atoms coupled to a radiation field has most frequently been based on a rate equation, or energy balance approach.[1,2] While this approach is satisfactory in many situations, the rate equations ignore phase effects between the dipole moment of the radiators and the radiation field. As will be pointed out below, a more complete analysis can lead to interesting effects in lasers under certain conditions; in particular, to the extraction of coherent radiation from a population which is below inversion.

The model is essentially semiclassical, since the expectation values of products of field and atomic operators are factored into a product of field and atom expectation values. The results of this factorization do not differ significantly from the electrodynamic treatment for these purposes, and it is of some interest to note that "superradiant" spontaneous emission,[3] including the correct Einstein B coefficient, follows naturally from the resulting equations.

Phonomenological terms are added to the basic set of equations, in a way familiar from the field of magnetic resonance, to describe the loss of radiation from the resonant structure, the "dephasing" of the total dipole moment and the effect of pumping radiation. The equations are cast into a form amenable to numerical solution and a three-level, ruby prototype laser is discussed under various conditions such as Q spoiling[4] and variation of the material parameters.

* Present address: Physics Department, University of California, Riverside, California.

[1] H. Statz and G. A. deMars in *Quantum Electronics*, edited by C. H. Townes (Columbia University Press, New York, 1960).

[2] A. A. Vuylsteke, J. Appl. Phys. **34**, 1615 (1963).

[3] R. H. Dicke, Phys. Rev. **93**, 99 (1954).

[4] R. H. Hellwarth, *Advances in Quantum Electronics*, edited by J. R. Singer (Columbia University Press, New York, 1961).

II. MATHEMATICAL FORMULATION

Consider an N-atom system in which only two levels of each atom interact appreciably with the resonant modes of the electromagnetic (e.m.) field. In the dipole approximation, the Hamiltonian for the system, with zero-point energy of the e.m. field subtracted, is taken as

$$\mathcal{H}=\sum_j \hbar\Omega^j R_3{}^j+\sum_\lambda \hbar\omega_\lambda a_\lambda{}^+a_\lambda +\sum_{\lambda,j}\hbar K_\lambda{}^j(a_\lambda{}^+R_-{}^j+a_\lambda R_+{}^j). \quad \text{(II.1)}$$

The operators $R_3{}^j$, $R_\pm{}^j$, following Dicke's notation,[3] in the space of the jth atom satisfy the commutation rules

$$[R_3{}^j, R_\pm{}^{j'}]=\pm R_\pm{}^j\delta_{jj'}, \quad \text{(II.2a)}$$

$$[R_\pm{}^j, R_\mp{}^{j'}]=2R_3{}^j\delta_{jj'}, \quad \text{(II.2b)}$$

respectively, and the field creation and destruction operators $a_\lambda{}^+$ and a_λ of the λth mode satisfy the commutation rules

$$[a_\lambda, a_{\lambda'}{}^+]=\delta_{\lambda\lambda'}, \quad \text{(II.3a)}$$

$$[a_\lambda, a_{\lambda'}]=0. \quad \text{(II.3b)}$$

The electric and magnetic fields are assumed to have the normal mode expansions

$$\mathbf{E}(\mathbf{x},t)=-\sum_\lambda(2\pi\hbar\omega_\lambda)^{1/2}E_\lambda(\mathbf{x})\boldsymbol{\varepsilon}_\lambda(a_\lambda{}^++a_\lambda), \quad \text{(II.4a)}$$

$$\mathbf{H}(\mathbf{x},t)=-i\sum_\lambda(2\pi\hbar\omega_\lambda)^{1/2}H_\lambda(\mathbf{x})\mathbf{h}_\lambda(a_\lambda{}^+-a_\lambda). \quad \text{(II.4b)}$$

The $E_\lambda(\mathbf{x})$ and $H_\lambda(\mathbf{x})$ are normalized so that

$$(\boldsymbol{\varepsilon}_\lambda\cdot\boldsymbol{\varepsilon}_\lambda{}')\int E_\lambda(\mathbf{x})E_\lambda{}'(\mathbf{x})d(\text{vol})=\delta_{\lambda\lambda}{}', \quad \text{(II.5a)}$$

$$(\mathbf{h}_\lambda\cdot\mathbf{h}_\lambda{}')\int H_\lambda(\mathbf{x})H_\lambda{}'(\mathbf{x})d(\text{vol})=\delta_{\lambda\lambda}{}'. \quad \text{(II.5b)}$$

The electric dipole moment operator for the jth atom is taken as the odd operator

$$\boldsymbol{\mu}_{\text{op}}{}^j=\boldsymbol{\mu}^j(R_+{}^j+R_-{}^j)\delta(\mathbf{x}-\mathbf{x}^j). \quad \text{(II.6)}$$

We have defined the coupling constant

$$K_\lambda{}^j=(2\pi\omega_\lambda/\hbar)^{1/2}\boldsymbol{\mu}^j\cdot\boldsymbol{\varepsilon}_\lambda E_\lambda(\mathbf{x})\,, \tag{II.7}$$

and have also defined the zero of energy as midway between the two levels of each atom so that $E_2{}^j-E_1{}^j=\hbar\Omega^j$ and $E_2{}^j+E_1{}^j=0$, where $E_2{}^j$ and $E_1{}^j$ are the energies of the upper and lower state of the jth atom, respectively.

The operator identity valid for any time-independent operator O

$$i\hbar\dot{O}=[O,\mathcal{H}] \tag{II.8}$$

is easily worked out and gives the set

$$i\dot{R}_+{}^j=-\Omega^jR_+{}^j+2\sum_\lambda K_\lambda{}^ja_\lambda{}^+R_3{}^j\,, \tag{II.9a}$$

$$i\dot{R}_-{}^j=+\Omega^jR_-{}^j-2\sum_\lambda K_\lambda{}^ja_\lambda R_3{}^j\,, \tag{II.9b}$$

$$i\dot{R}_3{}^j=-\sum_\lambda K_\lambda{}^j(a_\lambda{}^+R_-{}^j-a_\lambda R_+{}^j)\,, \tag{II.9c}$$

$$i\dot{a}_\lambda=\omega_\lambda a_\lambda+\sum_j K_\lambda{}^jR_-{}^j\,, \tag{II.9d}$$

$$i\dot{a}_\lambda{}^+=-\omega_\lambda a_\lambda{}^+-\sum_j K_\lambda{}^jR_+{}^j\,. \tag{II.9e}$$

Normally, expectation values of these operator identities are taken over the complete density matrix of the system (atoms+field). Taking expectation values on the right-hand side of (II.9a), (II.9b), and (II.9c) gives expectation values of products of atom and field operators, e.g., $\langle a_\lambda R_3{}^j\rangle$, etc. The problem is then essentially intractable, except in perturbation theory, for systems more complicated than a single atom interacting with a single mode. However, if we now assume that the density matrix of the system can be factored into the direct product

$$\rho=\rho_f\otimes\rho_a{}^1\otimes\rho_a{}^2\otimes\cdots\rho_a{}^N \tag{II.10}$$

of atom and field density matrices, terms such as $\langle a_\lambda R_3{}^j\rangle$ factor to the product of expectation values $\langle a_\lambda\rangle\langle R_3{}^j\rangle$. This is equivalent to phenomenological semiclassical theory, or the "S.C.F.A." (self-consistent field approximation) of Willis.[5]

It is convenient to define the slowly varying complex quantities A_λ, r^j by

$$\langle a_\lambda\rangle=A_\lambda e^{-i\omega_\lambda t}\,, \tag{II.11a}$$

$$\langle a_\lambda{}^+\rangle=A_\lambda{}^*e^{+i\omega_\lambda t}\,, \tag{II.11b}$$

$$\langle R_+{}^j\rangle=r^je^{i\Omega^jt}\,, \tag{II.11c}$$

$$\langle R_-{}^j\rangle=r^{j*}e^{-i\Omega^jt}\,. \tag{II.11d}$$

In terms of these new variables, we obtain from (II.9), after taking expectation values with respect to the factored density matrix (II.10), the set of five coupled equations

$$i\dot{A}_\lambda=\sum_j K_\lambda{}^jr^{j*}e^{i(\omega_\lambda-\Omega^j)t}\,, \tag{II.12a}$$

$$i\dot{r}^j=2\sum_\lambda K_\lambda{}^jA_\lambda{}^*e^{i(\omega_\lambda-\Omega^j)t}R_3{}^j\,, \tag{II.12b}$$

$$i\dot{R}_3{}^j=-\sum_\lambda K_\lambda{}^j(A_\lambda{}^*r^{j*}e^{i(\omega_\lambda-\Omega^j)t}-A_\lambda r^je^{-i(\omega_\lambda-\Omega^j)t})\,, \tag{II.12c}$$

as well as their complex conjugates (where $\langle R_3{}^j\rangle$ is written simply as $R_3{}^j$).

We digress somewhat now and consider a special case of (II.12) to point out some of Dicke's results using these equations. Suppose for the moment that $\Omega^j=\Omega$ and $K_\lambda{}^j=K_\lambda$, independent of j. That is, all atoms are identical, as well as being confined in a volume whose dimensions are small compared to a wavelength $\lambda\simeq2\pi c/\Omega$. We can then define

$$r=\sum_j r^j \tag{II.13a}$$

and

$$n=\sum_j R_3{}^j\,, \tag{II.13b}$$

in which special case we find the first integrals of (II.12).

$$\sum_\lambda |A_\lambda|^2+n=\text{const} \tag{II.14a}$$

and

$$|r|^2+n^2=\text{const}=R^2. \tag{II.14b}$$

(II.14a) is, of course, an expression of the conservation of energy. In analogy to Dicke's terminology, we would call (II.14b) the conservation of "cooperation." In a discussion of permanent dipole moments (spins) (II.14b) would be an expression of the conservation of total angular momentum. Here, however, we explicitly have in mind induced electric dipole moments. It is convenient to represent a radiation process geometrically in a 3-dimensional space labeling the coordinates (1,2,3) following Feynman.[6,7] According to (II.14) we can picture this simple radiation process as the motion of a vector $\mathbf{R}$ on the surface of a sphere in this space, (which, incidentally, is real space when considering permanent dipoles). The projection of $\mathbf{R}$ onto the (3) axis is the value of n, the population difference divided by two, and the projection of $\mathbf{R}$ onto the (1) axis gives $\mathrm{Re}(r)$ which is proportional to the expectation value of the induced dipole moment. The value of the induced dipole moment is zero when all atoms are exactly in their upper or lower states $n=\pm(N/2)$ and so we see that the maximum possible value of $|\mathbf{R}|=N/2$ (N is the total number of radiators), and it can have the (constant) values $(N/2)$, $(N/2)-1$, $\cdots0$, depending on initial conditions of the problem.

---

[5] C. R. Willis, Bull. Am. Phys. Soc. **9**, 4, 399 (1964).

[6] R. P. Feynman, F. L. Vernon, Jr., and R. W. Hellwarth, J. Appl. Phys. **28**, 49 (1957).

[7] Y. H. Pao, J. Opt. Soc. Am. **52**, 871 (1962).

To obtain spontaneous emission (using the same restrictive assumptions $\Omega^j=\Omega$, $K_\lambda{}^j=K_\lambda$), we simply perform a power-series expansion in powers of the coupling constant K_λ

$$A_\lambda(t)=A_\lambda{}^0(t)+K_\lambda A_\lambda{}^{(1)}(t)+\cdots,$$
$$r(t)=r^0(t)+K_\lambda r^{(1)}(t)+\cdots.$$

Inserting these into (II.12) and equating like powers of K_λ gives

$$i\dot{A}_\lambda{}^{(1)}(t)=K_\lambda r^*(0)e^{i(\omega_\lambda-\Omega)t}, \tag{II.15a}$$

$$r^0(t)=r^0(0)=r(0). \tag{II.15b}$$

$A_\lambda{}^0(t)=A_\lambda(0)=0$ for spontaneous emission. Integration of (II.9a) and insertion into (II.9c) leads to

$$\dot{n}=-|r(0)|^2\frac{d}{dt}\sum_\lambda K_\lambda{}^2\frac{\sin^2(\omega_\lambda-\Omega)t/2}{[(\omega_\lambda-\Omega)/2]^2}. \tag{II.16}$$

For spontaneous emission from the small volume into a large box of volume V, we find, in the usual way[8] (replacing the summation by an integral)

$$\dot{n}\simeq-2|r(0)|^2\rho(\Omega)K_\Omega{}^2. \tag{II.17}$$

Using

$$\rho(\Omega)=V\Omega^2/c^3 \quad\text{and}\quad \langle K_\Omega{}^2\rangle_{\text{sphere}}=2\pi\Omega\mu^2/3\hbar V \tag{II.18}$$

gives

$$\dot{n}=-|r(0)|^2/\tau=-(R^2-n^2(0))/\tau, \tag{II.19}$$

which is the desired result. Here τ^{-1} is the usual Einstein B coefficient

$$\tau^{-1}=\tfrac{4}{3}(\Omega^3\mu^2/\hbar c^3).$$

Suppose the system is started off with all atoms exactly in the upper state, $R^2=(N/2)^2$. Then, according to (II.19), nothing happens. This is a point of unstable equilibrium in the semiclassical theory. However, if $R^2=(N/2)^2$, but $N-1$ atoms are excited and one atom is in its ground state, then $n=(N/2)-1$, $\dot{n}=-(N-1)/\tau$, and the system emits "normally," that is, proportional to the number of excited atoms. But if we start off with $R^2=(N/2)^2$ and with half the atoms excited and half in their lower states, then $n(0)=0$ and $\dot{n}=-[(N/2)^2/\tau]$. The population then starts to decrease at a rate proportional to the square of the total number of atoms, and it is this situation that Dicke calls "superradiant" or "coherent" spontaneous emission. The situation is analogous to a physical pendulum which "radiates" at a very slow rate initially when aligned closely along the (3) axis but rather much more rapidly when in the (1–2) plane (when the electric field then sees a large dipole moment).

We now return to consideration of the more general set of Eqs. (II.12). Interest is in casting these into a form describing an idealized laser operating in one mode; the resulting set of 4 nonlinear coupled equation will be cast in a form which will readily yield to numerical integration.

Define the new variables m_λ, ψ_λ so that

$$m_\lambda e^{i\psi_\lambda}=\sum_j K_\lambda{}^j r^{j*}e^{i(\omega_\lambda-\Omega^j)t}, \tag{II.20}$$

which is the (complex) dipole moment which couples into the λth mode; also define the e.m. field amplitude and phase e_λ and ϕ_λ of mode λ by

$$A_\lambda=e_\lambda e^{i\phi_\lambda}. \tag{II.21}$$

In terms of the new variables we have from (II.12)

$$\dot{e}_\lambda=-m_\lambda\sin(\phi_\lambda-\psi_\lambda), \tag{II.22a}$$

$$e_\lambda\dot{\phi}_\lambda=-m_\lambda\cos(\phi_\lambda-\psi_\lambda), \tag{II.22b}$$

$$\begin{aligned}\dot{m}_\lambda e^{i\psi_\lambda}+im_\lambda e^{i\psi_\lambda}\dot{\psi}_\lambda
&=\sum_{\lambda',j}2iK_\lambda{}^jK_{\lambda'}{}^je_{\lambda'}e^{i\phi_{\lambda'}}e^{i(\omega_\lambda-\omega_{\lambda'})t}R_3{}^j\\
&\quad+i\sum_j K_\lambda{}^jr^{j*}(\omega_\lambda-\Omega^j)e^{i(\omega_\lambda-\Omega^j)t}.\end{aligned} \tag{II.22c}$$

We now make the reasonable assumption that if we take an average of these equations assuming a random distribution of atomic sites $\mathbf{x}^j$, the modes will decouple due to the orthogonality condition (II.5a). That is, we now assume that

$$\begin{aligned}\langle K_\lambda{}^jK_{\lambda'}{}^j\rangle_{\text{av}}&=(2\pi\omega_\lambda/\hbar V)\langle(\boldsymbol{\mu}\cdot\boldsymbol{\varepsilon}_\lambda)^2\rangle_{\text{av}}\delta_{\lambda\lambda'}\\
&\cong(2\pi\omega_\lambda/\hbar V)\mu_\lambda{}^2\delta_{\lambda\lambda'},\end{aligned} \tag{II.22d}$$

taking $\boldsymbol{\mu}^j\simeq\boldsymbol{\mu}$, for all j. μ_λ is the component of the dipole moment which couples into mode λ.

The second term on the right of (II.22c) would vanish if $\Omega^j=\Omega=\omega_\lambda$. It represents a dephasing of the total dipole moment due to different local environments giving rise to a spread in natural frequencies about Ω. We wish to take this term into account in a way familiar from nuclear resonance theory by saying that when $e_\lambda=0$, its effect is to cause the magnitude m_λ to decay exponentially to zero at a rate γ_λ. Then we write

$$\dot{m}_\lambda+\gamma_\lambda m_\lambda=-2\Gamma e_\lambda n\sin(\phi_\lambda-\psi_\lambda), \tag{II.23}$$

$$\dot{\psi}_\lambda m_\lambda=2\Gamma e_\lambda n\cos(\phi_\lambda-\psi_\lambda), \tag{II.24}$$

with

$$n=\sum_j R_3{}^j \tag{II.25}$$

and

$$\Gamma=2\pi\omega_\lambda\mu_\lambda{}^2/\hbar V. \tag{II.26}$$

From (II.12c), (II.20), and (II.21), we have that

$$\dot{n}=e_\lambda m_\lambda\sin(\phi_\lambda-\psi_\lambda)+\sum_{\lambda'\neq\lambda}e_{\lambda'}m_{\lambda'}\sin(\phi_{\lambda'}-\psi_{\lambda'}). \tag{II.27}$$

The sum $\sum_{\lambda'}$ runs over all those modes, except mode λ, which are at approximately frequency ω_λ but which are appreciably more lossy than the mode designated by λ. For instance, this term represents the losses of

[8] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955).

radiation from the sides of a lasing crystal due to spontaneous emission. The usually small stimulated emission into these modes which are characterized by high loss (free-space-like modes) will be neglected.

If we add a phenomenological mode loss rate term to (II.22a), and collect equations, we have finally the set

$$(d/dt)(e_\lambda^2)+2\beta e_\lambda^2=-m_\lambda e_\lambda \sin\theta_\lambda\,, \tag{II.28a}$$

$$(d/dt)m_\lambda+\gamma_\lambda m_\lambda=-2\Gamma e_\lambda n \sin\theta_\lambda\,, \tag{II.28b}$$

$$(d/dt)n=e_\lambda m_\lambda \sin\theta_\lambda+\sum_{\lambda'} e_{\lambda'} m_{\lambda'} \sin\theta_{\lambda'}\,, \tag{II.28c}$$

$$m_\lambda(d/dt)\psi_\lambda=2\Gamma e_\lambda n \cos\theta_\lambda\,, \tag{II.28d}$$

$$e_\lambda(d/dt)\phi_\lambda=-m_\lambda \cos\theta_\lambda\,, \tag{II.28e}$$

$$\theta_\lambda\equiv\phi_\lambda-\psi_\lambda\,. \tag{II.29}$$

From (II.28a), (II.28d), and (II.28e) we note the first integral

$$e_\lambda m_\lambda \cos\theta_\lambda=(e_\lambda m_\lambda \cos\theta_\lambda)_0 e^{-(\beta_\lambda+\gamma_\lambda)t}\,, \tag{II.30}$$

which states that if m_λ, e_λ, or $\cos\theta_\lambda$ were ever zero, then $\cos\theta_\lambda=0$ for all time for a nontrivial solution to exist. Otherwise, $\cos\theta_\lambda$ will approach zero at a rate $\beta_\lambda+\gamma_\lambda$. $\operatorname{Cos}\theta_\lambda=0$ implies $\sin\theta_\lambda=\pm1$. It must be noticed, however, that $\sin\theta_\lambda$ as a function of time takes on the two values $+1$ or -1, since by inspection of (II.28a), (II.28c) the sign of $\sin\theta_\lambda$ determines whether the atomic system is emitting or absorbing radiation.

If in (II.28b) we assume that $\dot{m}_\lambda \ll K_\lambda m_\lambda$, then we may neglect $\dot{m}_\lambda$, solve for m_λ, and insert its value into (II.28a) and (II.28c). This gives, by use of (II.30) with $\cos\theta_\lambda=0$,

$$(d/dt)e_\lambda^2+2\beta_\lambda e_\lambda^2=(2\Gamma/\gamma_\lambda)e_\lambda^2 n\,, \tag{II.31a}$$

$$(d/dt)n=-(2\Gamma/\gamma_\lambda)e_\lambda^2 n-(n+N/2)/\tau\,. \tag{II.31b}$$

The term $(n+N/2)/\tau$ has been used as an approximation for the last term of (II.28c); $n+(N/2)$ is simply the population of the excited state and τ^{-1} is the spontaneous rate. The two coupled equations constitute the well-known rate equation approach to radiation dynamics.[1] The absorption rate (per atom) for mode λ is given by $\alpha_\lambda=\Gamma/2\gamma_\lambda=\pi\omega_\lambda\mu_\lambda^2/\hbar V\gamma_\lambda$.

At the other extreme from the rate equation approximation (II.31), the completely "coherent" description of radiation into a single mode is afforded by dropping all terms in (II.19) which describe losses, that is by setting $\gamma_\lambda=\beta_\lambda=\sum e_{\lambda'}m_{\lambda'}\sin\theta_{\lambda'}=0$. In this case, by defining the angle variable Θ in the $m_\lambda/(2\Gamma)^{1/2}$, n plane as $\Theta=\tan^{-1}[(2\Gamma^{1/2})m_\lambda/n]$ we find the equation of motion for Θ

$$(d^2\Theta/dt^2)+(2\Gamma)\sin\Theta=0\,, \tag{II.32}$$

which is simply the equation of motion of a spherical pendulum. The initial conditions are determined from $d\Theta/dt=-(2\Gamma)^{1/2}e_{\lambda'}$. The solutions are elliptic functions and become sinusoids when $e_\lambda(0)$ gets large ($\gtrsim 10$). The case $e_\lambda(0)=0$ appropriate to spontaneous emission leads to the same general results as in Eq. (II.19) concerning the rate of spontaneouse mission as a function of N. Keeping only the mode loss rate in (II.28) results in the equation

$$(d^2\Theta/dt^2)+\beta_\lambda(d\Theta/dt)+2\Gamma\sin\Theta=0\,, \tag{II.33}$$

easily recognized as the equation for a damped physical pendulum.

The "perfectly coherent" process is one which describes a circle in the n, $m_\lambda/(2\Gamma)^{1/2}$ plane. The solutions to the rate equation, on the other hand, are restricted to one quadrant in this plane. Since $m_\lambda\simeq\Gamma e_\lambda n/\gamma_\lambda$, $n=0$ implies that $m_\lambda=0$, and thus the only crossing point of the axis $n=0$ is the improbable point $n=0$, $m_\lambda=0$, in the rate equation approximation.

In order to facilitate numerical integration of this set of equations, we define the new variables

$$I=e_\lambda^2/2N\,, \tag{II.34a}$$

$$m=m_\lambda/N(2\Gamma)^{1/2}\,, \tag{II.34b}$$

$$\zeta=-me_\lambda \sin\theta_\lambda/(2N)^{1/2}\,, \tag{II.34c}$$

$$\Delta=n/N\,, \tag{II.34d}$$

$$\alpha=(N\Gamma)^{1/2}\,. \tag{II.34e}$$

The equations (II.28) then take the form

$$\dot{I}+2\beta I=\alpha\zeta\,, \tag{II.35a}$$

$$dm^2/dt+2\gamma m^2=2\alpha\zeta\Delta\,, \tag{II.35b}$$

$$d\zeta/dt+(\beta+\gamma)\zeta=\alpha(m^2+2I\Delta)\,, \tag{II.35c}$$

$$\dot{\Delta}=-\alpha\zeta-[(\Delta+\tfrac{1}{2})/\tau]-P(\Delta-\tfrac{1}{2})\,, \tag{II.35d}$$

where we have suppressed the subscript λ, written the term $\sum_{\lambda'} e_{\lambda'}m_{\lambda'}\sin\theta_{\lambda'}$ as $(\Delta+\frac{1}{2})/\tau$, and added a term to the equation for the rate of change of the population difference Δ (normalized to $2N$) to describe the effect of a "pump" which takes atoms from the ground state and puts them into the upper state. We are thus assuming an infinitely fast transition from the pumping band to the upper laser level, as well as an optically thin material so that the pumping is uniform throughout the volume V. We can allow P (the pumping rate) as well as β, the cavity mode loss rate, to be functions of time; the numerical solutions below, however, only include the case of a time varying β.

III. RESULTS OF NUMERICAL INTEGRATION

The set of Eqs. (II.35) were integrated on a high-speed digital computer for several values of a (constant) pump rate P, and the coupling parameter α, and the dipole moment dephasing rate γ. The function β is taken to be a "Fermi function" of time

$$\beta(t)=[(\beta_i-\beta_f)/(1+e^{(t-t_0)/\Delta t})]+\beta_f\,, \tag{III.1}$$

and will represent approximately the "Q switching" by a Kerr cell or similar apparatus. t_0 will be chosen as a

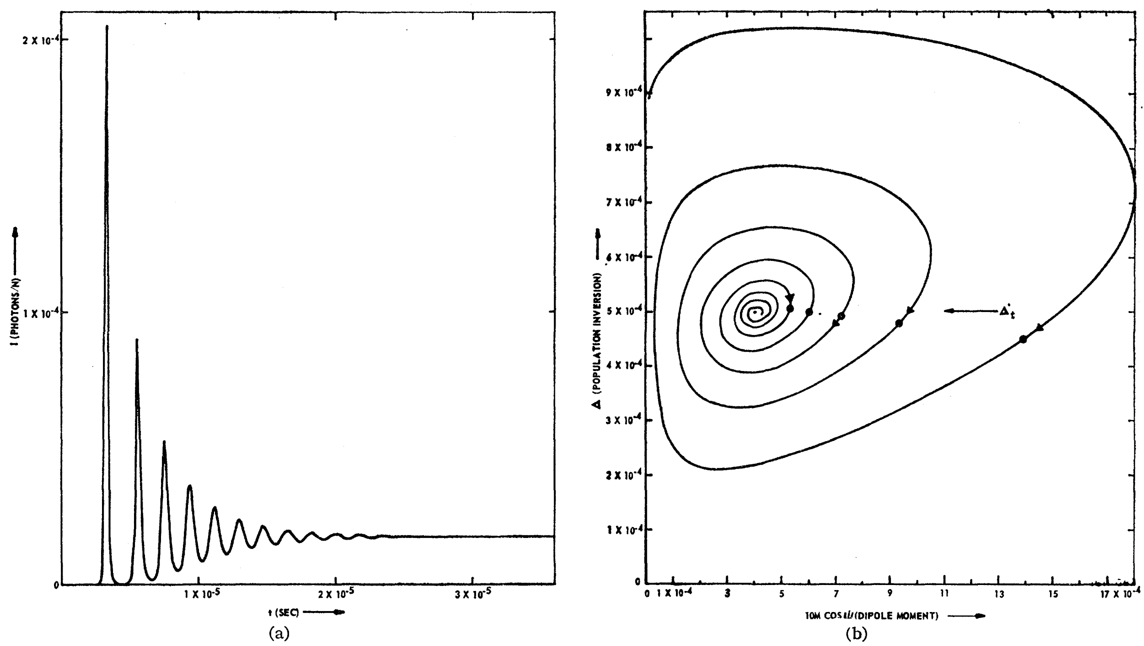


FIG. 1. Laser behavior with constant pump and constant cavity loss rate under conditions which would be well approximated by the rate equation approach. Parameter values are $\beta=10^7$, $\gamma=10^8$, $P=10^3$, $\alpha=10^9$ and $\tau^{-1}=3\times10^2$, all in sec^{-1}. $t=0$ is chosen from computed data near the start of significant behavior. (a) Intensity dependence on time. (b) Dipole moment versus population inversion.

time somewhere just before the populations Δ reach a threshold determined by β_i.

Figure 1 shows the "relaxation oscillations" for a laser operating under conditions of ordinary power output.[9] Both the intensity I as a function of time as well as the corresponding phase diagram of ($m\cos\psi$) versus Δ is shown for each case. The dots in the phase diagrams correspond in time to the peaks in the intensity diagram. In Fig. 1(b), the entire phase diagram is seen to lie in the first quadrant and the solution is very well predicted by the rate equations, since the rate γ is much greater than the fractional rate of change of m. The intensity I is shown in units of photons/N, where N is the effective number of atoms participating. By changing the cavity loss rate to a larger value we see in Fig. 2, as expected, that the

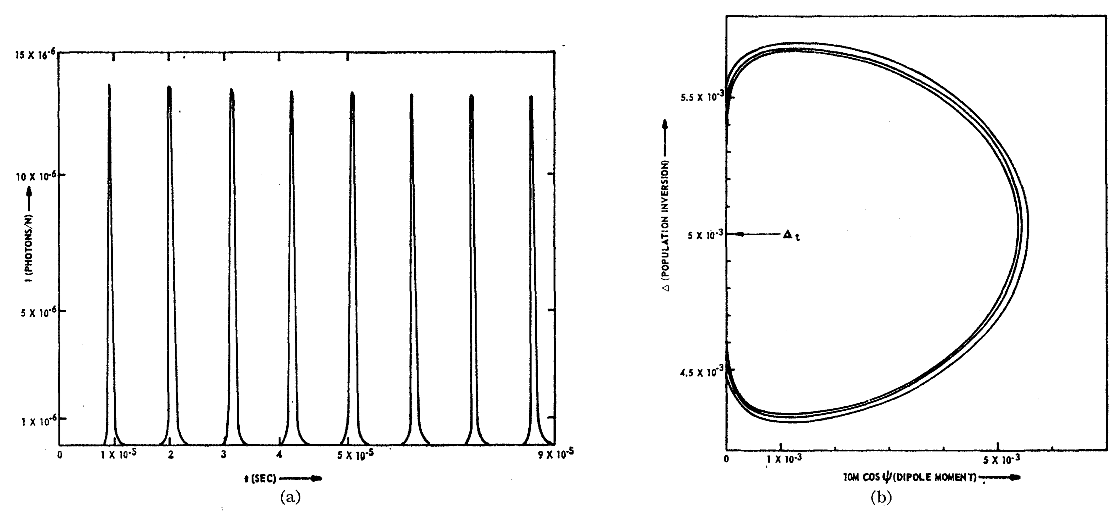
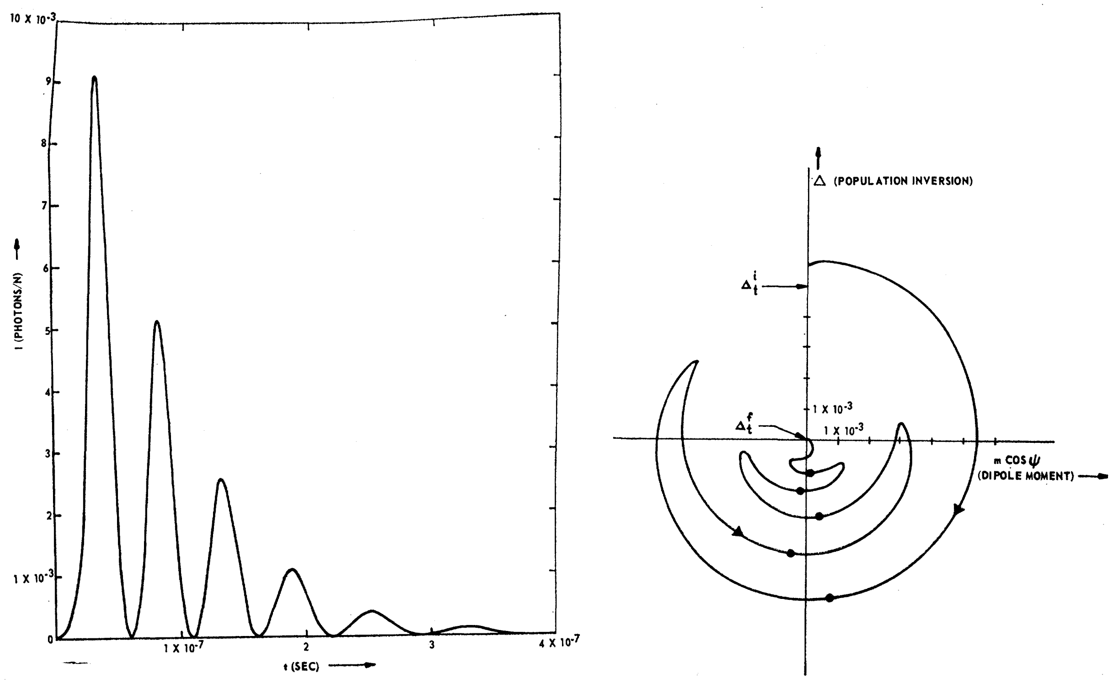


FIG. 2. Same as Fig. 1, except for higher cavity loss rate, $\beta=10^8$ sec^{-1}.

[9] For all the following figures, the quantity $(e_\lambda m_\lambda \cos\theta_\lambda)_0$ was taken as zero. [See Eq. (II.30).] Moreover, for convenience in graphing, the same discontinuous properties are attributed to ψ and ϕ as are exhibited by $\theta=\phi-\psi$. That is, $\sin\phi$ and $\sin\psi$ as functions of time are made to alternate between the values $+1$ and -1 in such a manner that $\sin\theta(t)$ takes on the correct values and the functions $m\cos\psi$ and $e\sin\phi$ are continuous.



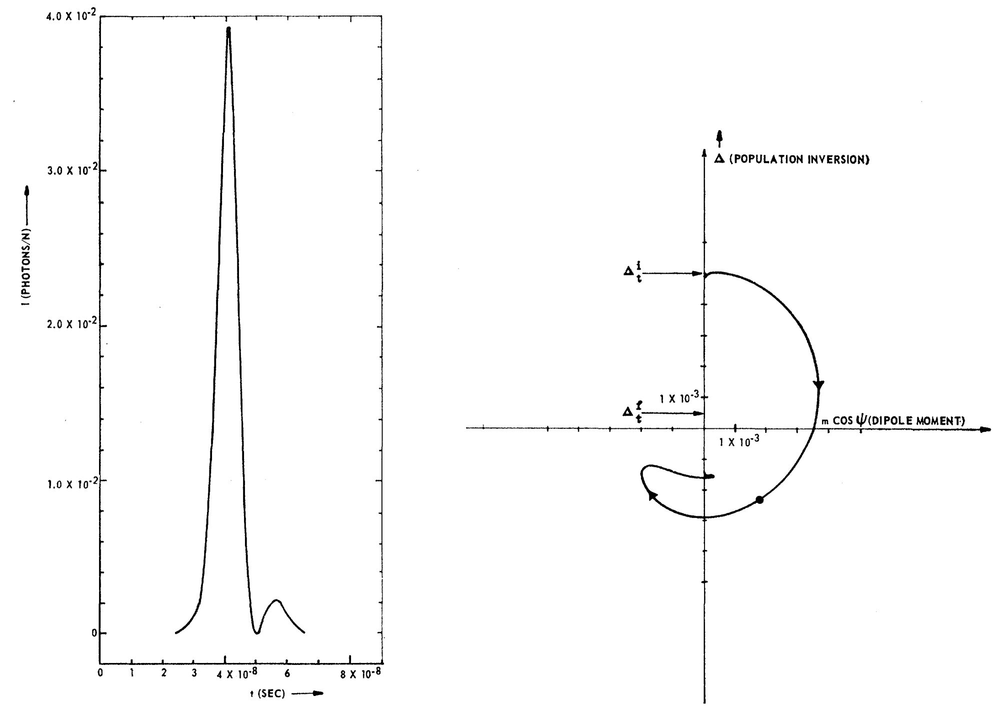
(a) (b)

FIG. 3. Same as Fig. 1, except for a larger pump $P=10^4$ sec^{-1}, a smaller dephasing rate $\gamma=10^7$ sec^{-1}, and the addition of Q spoiling with cavity loss rate initially $\beta_i=10^9$ sec^{-1} changing to a final cavity loss rate of $\beta_f=10^7$ sec^{-1} in a time $\Delta t=2.5.10^{-8}$ sec.

spiking is almost continuous; clearly the radiation almost completely escapes from the cavity before the pump can again bring the population over threshold, $\Delta_t=\beta\gamma/2\alpha^2$, and the process simply repeats itself. (The parameter τ, the spontaneous emission time has been taken as 3×10^{-3} sec in all computations.) Figure 3 shows a case of "Q switching" [Eq. (III.1)], where parameters are chosen to exhibit a rather extreme situation (in present day practice) of very low γ and β.

The second, third, etc., spikes in Fig. 3(a) are seen

FIG. 4. Same as Fig. 3, except that final cavity loss rate $\beta_f=10^8$ sec^{-1} and Q spoiling occurs over time $\Delta t=5.10^{-9}$ sec.



(a) (b)

to occur when the populations are below inversion, $\Delta<0$. This is strictly a possibility inherent in this analysis and not in a rate equation description. Figure 4 again exhibits the same phenomena; the second spike occurs totally when $\Delta<0$.

In both Figs. 3(b) and 4(b) the initial and final threshold values $\Delta_t^{(i)}$ and $\Delta_t^{(f)}$ are shown. Notice the difference in scale between Figs. 1, 2 and 3, 4. Figs. 1(b) and 2(b) would appear as diffuse dots near the Δ axis in Figs. 3(b) and 4(b). A case has also been run under "normal" circumstances (no Q spoiling) with a low threshold ($\Delta_t \approx 0.001$) in which the output I versus t appeared as a series of almost random spikes. Apparently, a great variety of solutions is possible here as compared to the rate equation approach.

These results are intended to serve as illustrative of some possibilities in coherent device dynamics. The case of a "two-level" device akin to a ruby laser has guided the discussion. However, one expects that the laser line in ruby is homogeneously broadened, and at room temperatures the parameter γ would be $\approx 10^{11}$ cps, the full time width of the R_1 transition. If this is the case then the rate equations would be expected to as adequately describe the dynamics as the full set, (II.35) and secondary pulses as described above would not be possible unless one operated as a much lower temperature, $\lesssim 70°$K. However, if the laser line is inhomogeneously broadened even at room temperature, (which appears unlikely in ruby), then such "hopping" might occur, since γ would be expected to be a good bit smaller. The above set of equations should approximately describe the dynamics of such a system where γ would be determined by the cavity parameter β and the effective number of atoms participating in the radiation process into a single mode would be roughly $(N)\mathrm{eff} \approx (\Delta\omega_c/\Delta\omega)N$, where $\Delta\omega_c$ is the cavity linewidth (presuming this is what governs the linewidth in this case) and $\Delta\omega$ is the full laser atomic linewidth.

It may be pertinent to point out that one cannot expect to see such "superradiant" hopping effects by "Q spoiling" a gas laser such as the helium-neon, 4-level laser of Javan.[10] The point is simply that due to the very rapid emptying of the terminal laser state into a metastable state compared to the time width of a giant pulse, one can never achieve a condition of $\Delta<0$ with any appreciable population in the lower laser state; that is, these equations do not describe such a device even over the time interval of a giant pulse.

ACKNOWLEDGMENTS

The authors would like to express their appreciation to Paul M. Sutton, Sylvan Katz, M. H. Johnson, and M. B. White for valuable discussions.

[10] A. Javan, W. R. Bennett, Jr., and D. R. Herriot, Phys. Rev. Letters 6, 106 (1961).